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THE STABILITY OF A COMPRESSED ROTATING ROD

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The classical stability problem of a compressed hinged elastic rod rotating with constant angular velocity about the axis that passes through the hinges is considered. It is assumed that the compressive force is constant and the line of its action coincides with the axis of rotation of the rod. The stability of a solution of the nonlinear problem that describes deformation of the rod under the action of the compressive force and the distributed centrifugal load is studied within the framework of the stability theory of dynamic systems with distributed parameters. The buckling parameters of the problem are determined. Calculation results are given.

1. Formulation of the Problem. We consider a simply supported elastic rod compressed by the constant force P which rotates with constant angular velocity ω (Fig. 1). Before loading, the axis of the rod is a straight line which coincides with the x axis. After buckling, the curved rod is subjected to the force P and the distributed centrifugal load $q(x) = m(x)\omega^2 y(x)$, where $m(x)$ is the mass per unit length of the rod and $y(x)$ is the deflection of the loaded rod. It is assumed that the line of action of the force passes through the hinges. It is required to determine the conditions under which the rod buckles.

A part of the curved rod (Fig. 2) is in equilibrium if the sum of the moments of all the forces that act on this part of the rod about an arbitrary point vanishes:

$$M + M_P + M_q = 0. \tag{1.1}$$

Here M is the internal bending moment and M_P and M_q are the moments of the force P and the distributed load q, respectively. These moments are related to the deflection by the formulas [1]

$$M = EI\rho(x)\,\frac{d^2y}{dx^2}\Big[1 + \Big(\frac{dy}{dx}\Big)^2\Big]^{-1.5}, \qquad M_P = Py,$$

$$M_q = \frac{x}{l}\int\limits_0^l \Big[\int\limits_0^{x_1} q(x_2)\,dx_2\Big]dx_1 - \int\limits_0^x \Big[\int\limits_0^{x_1} q(x_2)\,dx_2\Big]dx_1,$$

in which l is the length of the rod, E is the modulus of elasticity, I is the cross-sectional moment of inertia, $\rho(x)$ is a function that specifies the variation of the bending rigidity along the rod, and x_1 and x_2 are the integration variables. The equation of bending (1.1) can be combined with the expressions for the moments to give

$$EI\rho(x)\,\frac{d^2y}{dx^2}\Big[1 + \Big(\frac{dy}{dx}\Big)^2\Big]^{-1.5} + Py$$

$$+ \frac{x}{l}\int\limits_0^l \Big[\int\limits_0^{x_1} q(x_2)\,dx_2\Big]dx_1 - \int\limits_0^x \Big[\int\limits_0^{x_1} q(x_2)\,dx_2\Big]dx_1 = 0. \tag{1.2}$$

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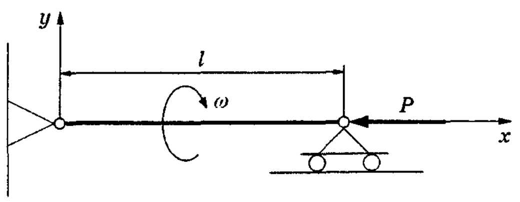

Fig. 1

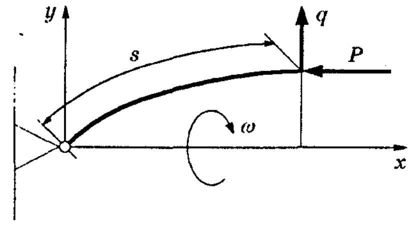

Fig. 2

In a rigorous nonlinear formulation, the boundary conditions are formulated with allowance for the displacement of the right end of the rod $\Delta = \int_0^l \left(\frac{dy}{dx}\right)^2 dx$ because of its bending (the rod is assumed to be incompressible):

$$y(0) = y(l - \Delta) = 0. \tag{1.3}$$

The conditions at the right end of the rod depend on the solution of the problem: hence, Eqs. (1.2) and (1.3) must be regarded as a system of coupled equations which can be solved only by numerical methods. In using analytical methods of solution, it is expedient to pass from the coordinate system (x, y) to the system (s, y), where s is the coordinate reckoned along the curved axis of the rod (Fig. 2). The coordinates x and s are related by the well-known relation between the differentiation operators:

$$\frac{d}{dx} = \left[1 - \left(\frac{dy}{ds}\right)^2\right]^{-0.5} \frac{d}{ds}. \tag{1.4}$$

Before passing to the new coordinate system, we reduce the integro-differential equation (1.2) to an equivalent differential equation. Differentiating Eq. (1.2) twice with respect to x, we obtain

$$\frac{d^2}{dx^2}\left[EI\rho(x)\frac{d^2y}{dx^2}\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{-1.5}\right] + P\frac{d^2y}{dx^2} - m(x)\omega^2 y = 0. \tag{1.5}$$

Using the differentiation operator (1.4), we determine the derivatives

$$\frac{dy}{dx} = \left[1 - \left(\frac{dy}{ds}\right)^2\right]^{-0.5}\frac{dy}{ds}, \qquad \frac{d^2y}{dx^2} = \left[1 - \left(\frac{dy}{ds}\right)^2\right]^{-2}\frac{d^2y}{ds^2}$$

and the moment

$$M = EI\rho(s)\frac{d^2y}{ds^2}\left[1 - \left(\frac{dy}{ds}\right)^2\right]^{-0.5},$$

which can be written in the form of power series:

$$\frac{dy}{dx} = \frac{dy}{ds}\left[1 + \frac{1}{2}\left(\frac{dy}{ds}\right)^2 + \frac{3}{8}\left(\frac{dy}{ds}\right)^4 + O\left(\left|\frac{dy}{ds}\right|^6\right)\right],$$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{ds^2}\left[1 + 2\left(\frac{dy}{ds}\right)^2 + 3\left(\frac{dy}{ds}\right)^4 + O\left(\left|\frac{dy}{ds}\right|^6\right)\right],$$

$$M = EI\rho(s)\frac{d^2y}{ds^2}\left[1 + \frac{1}{2}\left(\frac{dy}{ds}\right)^2 + \frac{3}{8}\left(\frac{dy}{ds}\right)^4 + O\left(\left|\frac{dy}{ds}\right|^6\right)\right].$$

Applying the differentiation operator (1.4) to the last equation twice, we obtain

$$\frac{d^2M}{dx^2} = EI\left[\rho(s)\frac{d^4y}{ds^4} + 2\frac{d\rho(s)}{ds}\frac{d^3y}{ds^3} + \frac{d^2\rho(s)}{ds^2}\frac{d^2y}{ds^2}\right.$$

$$+ \rho(s)\left(\frac{3}{2}\frac{d^4y}{ds^4}\left(\frac{dy}{ds}\right)^2 + 4\frac{dy}{ds}\frac{d^2y}{ds^2}\frac{d^3y}{ds^3} + \left(\frac{d^2y}{ds^2}\right)^3\right)$$

$$\left. + 3\frac{d\rho(s)}{ds}\left(\frac{d^3y}{ds^3}\left(\frac{dy}{ds}\right)^2 + \frac{dy}{ds}\left(\frac{d^2y}{ds^2}\right)^2\right) + \frac{3}{2}\frac{d^2\rho(s)}{ds^2}\frac{d^2y}{ds^2}\left(\frac{dy}{ds}\right)^2 + O(|y|^5)\right].$$

The resulting relations allow one to write Eq. (1.5) in the form of a sum comprising the linear operator

$$Ly = \rho(s)\,\frac{d^4y}{ds^4} + 2\,\frac{d\rho(s)}{ds}\,\frac{d^3y}{ds^3} + \left(\frac{d^2\rho(s)}{ds^2} + k^2\right)\frac{d^2y}{ds^2} - m(s)\omega_1^2 y,$$

the nonlinear operator

$$Ny = \rho(s)\left(\frac{3}{2}\,\frac{d^4y}{ds^4}\left(\frac{dy}{ds}\right)^2 + 4\,\frac{dy}{ds}\,\frac{d^2y}{ds^2}\,\frac{d^3y}{ds^3} + \left(\frac{d^2y}{ds^2}\right)^3\right)$$

$$+\,3\,\frac{d\rho(s)}{ds}\left(\frac{d^3y}{ds^3}\left(\frac{dy}{ds}\right)^2 + \frac{dy}{ds}\left(\frac{d^2y}{ds^2}\right)^2\right) + \left(\frac{3}{2}\,\frac{d^2\rho(s)}{ds^2} + 2k^2\right)\frac{d^2y}{ds^2}\left(\frac{dy}{ds}\right)^2,$$

and higher-order terms:

$$Ly + Ny + O(\,|y|^5\,) = 0. \tag{1.6}$$

In the expressions for the operators Ly and Ny, the following notation is used: $k^2 = P/(EI)$ and $\omega_1^2 = \omega^2/(EI)$.

Equations (1.6) must be supplemented by four boundary conditions (two boundary conditions at the points $s = 0$ and $s = l$). As applied to the rod considered, these conditions require that the deflections and the internal bending moments at the ends of the rod vanish:

$$y(0) = y(l) = 0, \qquad \frac{d^2y(0)}{ds^2} = \frac{d^2y(l)}{ds^2} = 0. \tag{1.7}$$

From the viewpoint of the general mathematical theory [2, 3], the stability analysis of the solution of the nonlinear problem (1.6), (1.7) is similar to the stability analysis of an immovable rod compressed by a constant force [4]. First, the spectral problem is solved and the space of eigenfunctions of the generator Ly is constructed. Then, the amplitude is determined as a projection of the solution of Eq. (1.6) onto the eigenspace associated with a conjugate eigenvector that belongs to the maximum eigenvalue of the spectrum. Finally, a solution of problem (1.6), (1.7) is constructed in the form of a power series of the amplitude.

2. Solution of the Spectral Problem. The spectrum of the generator Ly consists of the eigenvalues σ_n^2 ($n = 1, 2, \ldots$) of the boundary-value problem

$$Ly + \sigma^2 y = 0, \qquad y(0) = y(l) = 0, \qquad \frac{d^2y(0)}{ds^2} = \frac{d^2y(l)}{ds^2} = 0. \tag{2.1}$$

The eigenfunctions y_n associated with the eigenvalues σ_n^2 ($n = 1, 2, \ldots$) and any linear combinations of these eigenfunctions are the solutions of problem (2.1). The scalar product is determined in the space of these functions; therefore, the concept of amplitude can be introduced.

To determine the amplitude and establish the solvability of problem (1.6), (1.7), it is necessary to construct a conjugate operator L^* that satisfies the Green formula [5]

$$\int\limits_0^l [y^*Ly - yL^*y^*]\,dx = L_+[y, y^*]\Big|_0^l.$$

Here y and y^* are any solutions of the direct and conjugate problems, respectively; $L_+[y, y^*]\Big|_0^l$ is the bilinear form of the functions y and y^* and their derivatives of up to the third order inclusively.

The differentiation form L^*y^* conjugate to the form Ly is given by

$$L^*y^* = \frac{d^4}{ds^4}[\rho(s)y^*] - \frac{d^3}{ds^3}\left[2\,\frac{d\rho(s)}{ds}\,y^*\right] + \frac{d^2}{ds^2}\left[\left(\frac{d^2\rho(s)}{ds^2} + k^2\right)y^*\right] - m(s)\omega_1^2 y^*.$$

Calculating the derivatives and combining the terms, we obtain

$$L^*y = \rho(s)\,\frac{d^4y^*}{ds^4} + 2\,\frac{d\rho(s)}{ds}\,\frac{d^3y^*}{ds^3} + \left(\frac{d^2\rho(s)}{ds^2} + k^2\right)\frac{d^2y^*}{ds^2} - m(s)\omega_1^2 y^*.$$

It follows that the operator L is self-conjugate: $L = L^*$. To determine the boundary conditions of the operator L^*, we write the right side of the Green formula in the bilinear form

$$L_+[y, y^*]\Big|_0^l = \Big\{\rho(s)y^* \frac{d^3y}{ds^3} - \Big[\rho(s)\frac{dy^*}{ds} - \frac{d\rho(s)}{ds}y^*\Big]\frac{d^2y}{ds^2}$$

$$+\Big[\rho(s)\frac{d^2y^*}{ds^2} + k^2y^*\Big]\frac{dy}{ds} - \Big[\rho(s)\frac{d^3y^*}{ds^3} + \frac{d\rho(s)}{ds}\frac{d^2y^*}{ds^2} + k^2\frac{dy^*}{ds}\Big]y\Big\}_0^l.$$

One can readily verify that $L_+[y, y^*]\Big|_0^l = 0$ for arbitrary dy/ds, d^3y/ds^3, dy^*/ds, and d^3y^*/ds^3 provided

$$y^*(0) = y^*(l) = 0, \qquad \frac{d^2y^*(0)}{ds^2} = \frac{d^2y^*(l)}{ds^2} = 0. \tag{2.2}$$

Comparing the boundary conditions (2.2) of the conjugate problem, for which the bilinear form of the Green formula vanishes, with conditions (1.7), we infer that problem (2.1) is self-conjugate for all continuous functions $\rho(s)$ and their derivatives of up to the third order inclusively. The fact that the problem is self-conjugate simplifies its solution. However, because of the variable coefficients $\rho(s)$ and $m(s)$, the analytical determination of the eigensolutions of problem (2.1) is a difficult mathematical problem. Therefore, below, without loss of generality we assume that the rod is characterized by the constant rigidity $\rho = 1$ and the constant specific mass $m = m_0$.

For constant coefficients, problem (2.1) admits the exact solution

$$y = c_1 \sin(\lambda s) + c_2 \cos(\lambda s) + c_3 s + c_4, \tag{2.3}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants. Solving the boundary-value problem (2.3), (2.1), we obtain the system

$$c_2 + c_4 = 0, \qquad c_1 \sin(\lambda l) + c_2 \cos(\lambda l) + c_3 l + c_4 = 0,$$

$$c_2 = 0, \qquad c_1 k^2 \sin(\lambda l) + c_2 k^2 \cos(\lambda l) = 0,$$

which implies that $c_2 = c_3 = c_4 = 0$. Problem (2.1) has a nontrivial solution when $c_1 \neq 0$ and the parameter λ takes on discrete values $\lambda_n = n\pi l^{-1}$ $(n = 1, 2, \ldots)$ associated with the eigenfunctions $y_n = c_1 \sin(\lambda_n s)$. The eigenfunctions are determined up to a constant factor; we set $c_1 = 1$ and use it as a coefficient with the dimension of length. The choice of the value of c_1 is motivated by the fact that within the framework of the method, the eigenfunctions y_n $(n = 1, 2, \ldots)$ are regarded merely as independent coordinates of a certain space of functions. In other words, the coefficient c_1 in the equation for rod deflection can take on different nonzero values.

The eigenfunctions of the operator L^* which are conjugate relative to the scalar product coincide with the eigenfunctions of the operator L with accuracy up to arbitrary factors A_n, since the problem is self-conjugate: $y_n^* = A_n y_n$. Substitution of the eigenfunctions y_n into the differential equation (2.1) yields the characteristic equation for the eigenvalues of the operator L:

$$\sigma_n^2 = \lambda_n^2 k^2 + m_0\omega_1^2 - \lambda_n^4 \qquad (n = 1, 2, \ldots). \tag{2.4}$$

It follows from Eqs. (2.4) that there exists an infinite set of values of the force P that correspond to curvilinear configurations of the rod. The minimum value of the axial force is determined by the eigenvalue σ_1^2.

We introduce the notation $\sigma_1^2 \doteq \mu\pi^2 l^{-2}$, where μ is a parameter from the zero-containing open interval. With allowance for the expressions for k^2 and ω_1^2, we write Eq. (2.4) for $n = 1$ in the form

$$\mu = P/(EI) + m_0\omega^2 l^2/(EI\pi^2) - (\pi/l)^2. \tag{2.5}$$

The solution of the linear problem (2.1) is stable for $\mu < 0$ and unstable for $\mu > 0$. In the linear approximation, the critical axial force is determined from the condition $\mu = 0$ as follows:

$$P^* = EI\pi^2/l^2 - m_0\omega^2 l^2/\pi^2. \tag{2.6}$$

For $\omega = 0$, relation (2.6) coincides with the well-known Euler formula [1]. Solution (2.6) of the linearized problem (2.1) coincides with the solution of the stability problem of a rod on an elastic foundation [6, 7] if the elastic-foundation coefficient is assumed to be negative.

One can also determine the critical velocity of rotation $\omega^* = \sqrt{EI\pi^4/(m_0 l^4) - P\pi^2/(m_0 l^2)}$. In the space of the independent coordinates P, l, and ω, the stability of the solutions of the boundary-value problem (2.1) is bounded by the surface $\mu(P, l, \omega) = 0$.

3. Stability of the Bifurcation Solution. It is well known that an immovable rod buckles if the load exceeds the critical value obtained from the solution of the linearized problem [6, 7]. It follows from the theory of nonlinear operator equations [8] that in the neighborhood of the bifurcation point of the solution of the linearized equation $\mu = 0$, the nonlinear equation (1.6) has small nontrivial solutions for $\mu < 0$ and $\mu > 0$. Since the operators Ly and Ny are smooth, the nontrivial solution is unique and contains information on the load and rod configurations after buckling.

Using the relation $k^2 = \mu - m_0\omega_1^2 l^2/\pi^2 + (\pi/l)^2$, which follows from (2.5), we rewrite the operators Ly and Ny in Eq. (1.6) with allowance for the fact that the rod is uniform:

$$L_\mu y = \frac{d^4 y}{ds^4} + \Big[\mu - \frac{m_0\omega_1^2 l^2}{\pi^2} + \Big(\frac{\pi}{l}\Big)^2\Big]\frac{d^2 y}{ds^2} - m_0\omega_1^2 y; \tag{3.1}$$

$$N_\mu y = \frac{3}{2}\frac{d^4 y}{ds^4}\Big(\frac{dy}{ds}\Big)^2 + 4\frac{dy}{ds}\frac{d^2 y}{ds^2}\frac{d^3 y}{ds^3} + \Big(\frac{d^2 y}{ds^2}\Big)^3$$

$$+ \frac{dy}{ds}\Big(\frac{d^2 y}{ds^2}\Big)^2 + 2\Big[\mu - \frac{m_0\omega_1^2 l^2}{\pi^2} + \Big(\frac{\pi}{l}\Big)^2\Big]\frac{d^2 y}{ds^2}\Big(\frac{dy}{ds}\Big)^2 = 0. \tag{3.2}$$

Bearing in mind (3.1) and (3.2), we write Eq. (1.6) in the form of a series in powers of y and μ in the neighborhood of the point $(y, \mu) = (0, 0)$:

$$L_0 y + \mu\frac{\partial L_0}{\partial \mu} y + N_0(y, y, y) + O(\,|y|^5\,) = 0, \tag{3.3}$$

where $L_0 y = L_{\mu=0} y$ and $N_0(y, y, y) = N_{\mu=0}(y, y, y)$. Now, we define the amplitude as a scalar product $\varepsilon = \int_0^l y y_1^*\, ds$ and seek a solution of Eq. (3.3) in the form of series in powers of ε:

$$y = \sum_{k=1}^{\infty}\frac{Y_k\varepsilon^k}{k!}, \qquad \mu = \sum_{k=1}^{\infty}\frac{\mu_k\varepsilon^k}{k!}, \tag{3.4}$$

where Y_k and μ_k are the expansion coefficients to be determined.

Substituting series (3.4) into Eq. (3.3) and collecting terms of like powers of ε up to the third power, we obtain the system

$$L_0 Y_1 = 0; \tag{3.5}$$

$$L_0 Y_2 + 2\mu_1\frac{\partial L_0}{\partial \mu} Y_1 = 0; \tag{3.6}$$

$$L_0 Y_3 + 3\mu_1\frac{\partial L_0}{\partial \mu} Y_2 + 3\mu_2\frac{\partial L_0}{\partial \mu} Y_1 + 6N_0(Y_1, Y_1, Y_1) = 0. \tag{3.7}$$

Equation (3.5) has the unique solution $Y_1 = y_1$. The general solution of the nonhomogeneous equation (3.6) is written as a sum of the solution of the homogeneous equation $L_0 Y_2 = 0$, which coincides with the solution of Eq. (3.5), and the particular solution of Eq. (3.6). The solvability condition of this equation

$$\int_0^l \mu_1 y_1^*\frac{\partial L_0}{\partial \mu} Y_1\, dx = -\frac{\mu_1\pi^2}{l^2}\int_0^l y_1 y_1^*\, dx = 0$$

holds only for $\mu_1 = 0$. It follows that the particular solution of Eq. (3.6) is zero.

The solvability condition of the nonhomogeneous equation (3.7)

$$\mu_2 \int_0^l y_1^* \frac{\partial L_0}{\partial \mu} y_1 \, dx + 2 \int_0^l y_1^* N_0(y_1, y_1, y_1) \, dx = 0$$

and the relation

$$\frac{\partial L_0}{\partial \mu} y_1 = -\frac{\pi^2}{l^2} y_1$$

can be combined to give

$$\mu_2 = \frac{2l^2}{\pi^2} \int_0^l y_1^* N_0(y_1, y_1, y_1) \, dx \left[\int_0^l y_1 y_1^* \, dx \right]^{-1}. \tag{3.8}$$

Substituting (3.8) into (3.4), we obtain the bifurcation solution

$$\mu = 0.5\mu_2 \varepsilon^2, \tag{3.9}$$

which determines the stability boundary of the solution of the nonlinear problem (1.6) and (1.7) in the plane (μ, ε). For the solution (3.9) be unique, it is necessary to use the normalization condition $\varepsilon = 1$, which follows from the definition of the amplitude [4]. This condition implies the formula for the coefficient

$$A_1 = \left[\int_0^l y_1^2 \, dx \right]^{-1} = \frac{2}{c_1 l}$$

in the expression for the conjugate eigenvector y_1^*.

Substitution of y_1, y_1^* into formula (3.8) yields an exact value of the parameter $\mu_2 = c_1^2\pi^4/(4l^4) + c_1^2 m_0 \omega_1^2$. Combining the last relation, Eq. (2.5), and the normalizing condition $\varepsilon = 1$, we write Eq. (3.9) in the form

$$\frac{P}{EI} + \frac{m_0\omega^2 l^2}{EI\pi^2} - \left(\frac{\pi}{l}\right)^2 = \frac{c_1^2\pi^4}{8l^4} + \frac{c_1^2 m_0 \omega^2}{2EI}. \tag{3.10}$$

Solution (3.10) is called a supercritical solution. It exists only for $\mu > 0$ [the left side of (3.10) is equal to μ and the right side is always positive] and is stable in the neighborhood of the point $\mu = 0$.

Equation (3.10) implies that the maximum load $P_{\max}$ at which the rod buckles has the form

$$P_{\max} = \frac{EI\pi^2}{l^2}\left(1 + \frac{c_1^2\pi^2}{8l^2}\right) - m_0\omega^2\left(\frac{l^2}{\pi^2} - \frac{c_1^2}{2}\right). \tag{3.11}$$

In the case where the rod of length $l = \pi$ is immovable ($\omega = 0$), we obtain $P_{\max} = 1.125EI$, which corresponds to the result obtained for a uniform rod in [4]. From (3.11), one can readily determine the limiting angular velocity at a certain load value, including $P = 0$. Figure 3 shows the maximum load $P_{\max}$ as a function of the parameter $l_1 = l\pi^{-1}$ and the angular velocity ω for the case where $m_0 = 0.1$ kg/m, $EI = 1$ N $\cdot$ m^2, and $c_1 = 1$ m. It follows from formula (3.11) and Fig. 3 that the rotating rod buckles for a smaller compressive force compared to the immovable rod; the rod compressed by the axial force buckles at a smaller angular velocity compared to the unloaded rod.

Astapov and Kornev [9] obtained a relation between the maximum load and the deflection of an immovable rod f_1. For $\omega = 0$ and $c_1 = f_1$, expression (3.11) coincides with the result of [9] with accuracy up to $\pi^6 l^{-6}$.

The rod configuration that corresponds to the maximum load is determined by the formula

$$y = Y_1\varepsilon + Y_{3p}\varepsilon^3/3!, \tag{3.12}$$

which follows from (3.4). The function Y_1 is the solution of the homogeneous problem (3.5). The function Y_{3p} is the particular solution of the nonhomogeneous problem (3.7). The solutions of the homogeneous problem (3.6) and the corresponding homogeneous problem (3.7) are allowed for by the first term in (3.12).

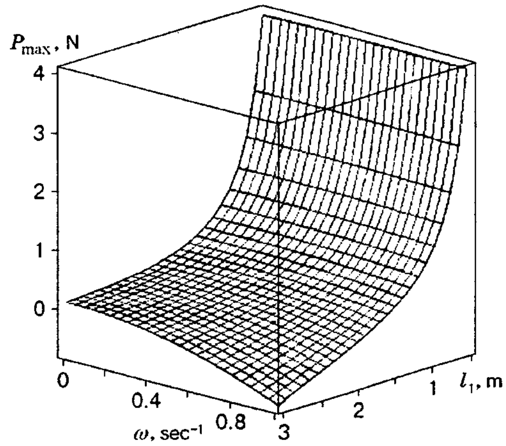

Fig. 3

For the known solution of the homogeneous equation $L_0 Y_3 = 0$ [which coincides with the solution of Eq. (3.5)], the particular solution of the nonhomogeneous equation (3.7) has the form [5]

$$Y_{3p} = \sin(\lambda_1 s) \int_0^s \Big[3\mu_2 \frac{\partial L_0}{\partial \mu} Y_1 + 6N_0(Y_1, Y_1, Y_1)\Big] \cos(\lambda_1 s)\, ds. \qquad (3.13)$$

The integral on the right side of (3.13) is expressed in terms of elementary functions; however, the resulting expression is cumbersome and is omitted here. To determine the deflection y at any point s, we substitute (3.13) into (3.12). Let, for example, $l = 2\pi$. Substituting this value into (3.11), we determine the maximum load $P_{max} = 0.258EI - 3.5m_0\omega^2$. The maximum deflection of the rod y_{max} occurs at the mid-span of the rod ($s = \pi$). Using π as the upper limit of integration on the right side of (3.13), we obtain $Y_{3p}(\pi) = 0.178 + 0.75m_0\omega^2/(EI)$. Further, assuming that $Y_1(\pi) = 1$, from (3.12) we obtain $y_{max} = \varepsilon + (0.0297 + 0.125m_0\omega^2/(EI))\varepsilon^3$. The normalization $\varepsilon = 1$ shows that the limit load corresponds to the maximum deflection of the rod $y_{max} = 1.0297 + 0.125m_0\omega^2/(EI)$. The following particular cases follow from the above results: for $\omega = 0$ and $l = 2\pi$, we have $P_{max} = 1.032P^*$ and $y_{max}/l = 0.164$; for the zero load, the rod of the same length buckles at the angular velocity $\omega = 1.179\omega^*$.

Calculating the derivatives dy/ds for $\omega = 0$ and $l = 2\pi$, we infer that at the ends of the rod, the tangent to the rod axis makes an angle of about 27° to the x axis. It follows that the calculation results for $\omega = 0$ and $l = 2\pi$ agree with the data of [9, Table 1].

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